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Techniques for Improved Convergence in Neighboring Optimum Guidance

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In the application of neighboring optimum feedback guidance schemes, the choice of the optimum reference state to compare with the perturbed state is not straightforward. Recent studies have shown that time-to-go is preferable to clock time and performance index-to-go as a lookup parameter. In this analysis the basic theory of neighboring optimum guidance is used to motivate a new lookup parameter called min-distance, which is determined by minimizing a suitable metric function of the perturbed state and the reference trajectory. This lookup parameter does not require an estimation of the perturbed final time whereas time-to-go requires such an estimate. A comparison of time-to-go and min-distance is simulated for Zermelo's problem, and it is shown that the neighborhood of convergence about the nominal trajectory is enlarged considerably with the min-distance lookup parameter technique.

I. Introduction

In recent years the idea of using a linear (and possibly higher-order) perturbation of a predetermined optimal trajectory for the feedback guidance of space vehicles has been advanced by a number of investigators.[1–5] The name most commonly associated with this approach is neighboring optimum guidance, and the fundamental problem that motivates the technique is the following. Let $\{x^*(t), u^*(t), \lambda^*(t), t\epsilon[\bar{t}_0,\bar{t}_f]\}$ denote a nonsingular optimal trajectory (the nominal) such that $J = \Phi(t_f,x_f)$ is minimized and the following conditions are satisfied:

$$x_i^*(\bar{t}_0) = x_{i0} \qquad (i = 1, \ldots, n) \tag{1}$$

$$N_i[\bar{t}_f,x^*(\bar{t}_f)] = 0 \qquad (i = 1, \ldots, p \leq n) \tag{2}$$

$$\dot{x}_i^*(t) = f_i[t,x^*(t),u^*(t)] \qquad (i = 1, \ldots, n) \tag{3}$$

or

$$\dot{x}_i^*(t) = f_i\{t,x^*(t),\mu[t,x^*(t),\lambda^*(t)]\} \; (i = 1, \ldots, n) \tag{4}$$

where

$$u_i(t) = \mu_i[t,x(t),\lambda(t)] \; (i = 1, \ldots, m) \tag{5}$$

are defined by the maximum principle. Let $(x_1, \ldots, x_n)$ be given. Determine a guidance program based on the nominal trajectory which transfers the vehicle from $(x_1, \ldots, x_n)$ to $N_i(t_f,x_f) = 0(i = 1, \ldots, p)$ while minimizing $J = \Phi(t_f,x_f)$.

In Refs. 1–4 techniques are developed to determine guidance functions of the form

$$u_i(t) = u_i^*(t) + \sum_{j=1}^{n} G_{ij}(\tau,t)[x_j - x_j^*(\tau)] \tag{6}$$

where $\tau\epsilon[\bar{t}_0,\bar{t}_f]$ is a parameter that associates the stored nominal values $u_i^*(t)$, $G_{ij}(\tau,t)$, $x_i^*(t)$ with the current state $(x_1, \ldots, x_n)$, i.e., a "lookup parameter." The time functions $G_{ij}(\tau,t)$ are the linear feedback gains for the guidance function. In Ref. 5 a technique is developed to determine the initial Lagrange multipliers, associated with the point $(x_1, \ldots, x_n)$, which can then be used to integrate the equations of motion and Euler-Lagrange equations. When the resultant solutions are substituted into Eqs. (5), the guidance function is determined.

In this paper the underlying theory of neighboring optimum guidance is used to motivate new techniques that enlarge the neighborhood of convergence about the nominal trajectory. The techniques are applied to a stationary problem and suggestions are given for nonstationary problems.

II. Theoretical Basis for Neighboring Optimum Guidance

In this section classical imbedding and implicit function theorems from the theory of differential equations will be used to define the range of applicability of neighboring optimum guidance. Portions of this section are just applications-oriented interpretations of Silber's excellent work.[5]

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Consider a nonsingular optimal trajectory problem for which the maximum principle has been applied to obtain the transversality conditions and controls as functions of the state, Lagrange multipliers, and time (in general). Then, the following equations must be satisfied on an optimal trajectory:

$$\dot{x}_i = f_i(t,x,\lambda) \quad (i = 1, \ldots, n) \tag{7}$$

$$\dot{\lambda}_i = g_i(t,x,\lambda) \quad (i = 1, \ldots, n) \tag{8}$$

$$x_i(t_0) = x_{i0} \quad (i = 1, \ldots, n) \tag{9}$$

$$M_i(t_f,x_f,\lambda_f) = 0 \quad (i = 1, \ldots, n+1) \tag{10}$$

where Eqs. (10) represent both the geometrical terminal constraints and the transversality conditions. We shall usually denote Eqs. (7–10) as vectors, e.g., $\dot{x} = f(t,x,\lambda)$, etc.

For a neighboring optimum guidance function to exist, the functions involved in Eqs. (7–10) must satisfy the following conditions.

Assumption 1 There exist real numbers $r_i > 0 (i = 0, \ldots, 2n)$ for each $(t,x,\lambda)\epsilon S_1 \equiv \{(t,x,\lambda)\}$ such that the vector functions $f(t,x,\lambda)$ and $g(t,x,\lambda)$ can be represented by convergent Taylor series expansions about the points $(\bar{t},\bar{x},\bar{\lambda})\epsilon S_1$ (i.e., S_1 is the set of points at which the functions are analytic) in a neighborhood N_1 of $(\bar{t},\bar{x},\bar{\lambda})$, where $N_1 \equiv \{|t - \bar{t}| < r_0, |x_1 - \bar{x}_1| < r_1, \ldots, |x_n - \bar{x}_n| < r_n, |\lambda_1 - \bar{\lambda}_1| < r_{n+1}, \ldots, |\lambda_n - \bar{\lambda}_n| < r_{2n}\}$. Also, there exist real numbers $R_i > 0$ $(i = 0, \ldots, 2n)$ for each $(t,x,\lambda)\epsilon S_2 \equiv \{(t,x,\lambda)\}$ such that the vector function $M(t_f,x_f,\lambda_f)$, when considered as a function of (t,x,λ), is analytic at each $(\bar{t},\bar{x},\bar{\lambda})\epsilon S_2$ where the Taylor series is valid in a neighborhood $N_2 \equiv \{|t - \bar{t}| < R_0, \ldots, |\lambda_n - \bar{\lambda}_n| < R_{2n}\}$.

The first step in the application of a neighboring optimum guidance scheme is the determination of a reference optimal trajectory, which we shall call the nominal trajectory. Actually, such a trajectory is a particular solution of Eqs. (7–10) that satisfies additional conditions.

Assumption 2 (The Nominal Trajectory) Let $x = \phi^*(t)$, $\lambda = \psi^*(t)$ represent a particular solution of Eqs. (7) and (8) that satisfies the boundary conditions Eqs. (9) and (10) on the interval $[\bar{t}_0,\bar{t}_f]$. Furthermore, assume that the particular solution satisfies the following conditions: a) $[t,\phi^*(t), \psi^*(t)]\epsilon S_1$ for each $t\epsilon[\bar{t}_0,\bar{t}_f]$; b) $[\bar{t}_f,\phi^*(\bar{t}_f),\psi^*(\bar{t}_f)]\epsilon S_2$; c) $M[\bar{t}_f, \phi^*(\bar{t}_f),\psi^*(\bar{t}_f)] = 0$; d) $M[t,\phi^*(t),\psi^*(t)] \neq 0$ for each $t\epsilon[\bar{t}_0,\bar{t}_f)$; and e) $[\partial(M_1^*, \ldots, M_{n+1}^*)]/\{\partial[\bar{t}_f,\psi_1^*(t), \ldots \psi_n^*(t)]\} \neq 0$ for each $t\epsilon[\bar{t}_0,\bar{t}_f)$.

Condition a) requires the functions $f(t,x,\lambda)$ and $g(t,x,\lambda)$ to be analytic at each point on the nominal; b) requires $M(t_f, x_f,\lambda_f)$ to be analytic at $(\bar{t}_f,\bar{x}_f,\bar{\lambda}_f)$; c) and d) require that the terminal conditions be satisfied once and only once on the nominal; and e) is a consequence of the implicit function theorem that guarantees the existence of the desired feedback guidance function.

There exists a close relationship between condition e) and the generalized Jacobi test of Ref. 6 (where the elements of a matrix $P - RQ^{-1}R^T$ must be finite in order that a neighboring optimum guidance function exist). To verify that condition e) is satisfied, one must show that the determinant of the $(n + 1 \times n + 1)$ matrix $[a_{i1}b_{ij}(t_0)]$, $i = 1, \ldots, n+1$; $j = 1, \ldots, n$, is nonzero at each $t_0\epsilon[\bar{t}_0,\bar{t}_f)$, where

$$a_{i1} = \left[\frac{\partial M_i}{\partial t_f} + \sum_{k=1}^{n}\left(\frac{\partial M_i}{\partial x_{kf}} f_k + \frac{\partial M_i}{\partial \lambda_{kf}} g_k\right)\right]_{[\bar{t}_f,x^*(\bar{t}_f),\lambda^*(\bar{t}_f)]}$$

$$b_{ij}(t_0) = \sum_{k=1}^{n}\left[\left.\frac{\partial M_i}{\partial x_{kf}}\right|^*_{\bar{t}_f} \left.\frac{\partial \phi_k(\bar{t}_f;t_0,x_0,\lambda_0)}{\partial \lambda_j(t_0)}\right|^* + \left.\frac{\partial M_i}{\partial \lambda_{kf}}\right|^*_{\bar{t}_f} \left.\frac{\partial \psi_k(\bar{t}_f;t_0,x_0,\lambda_0)}{\partial \lambda_j(t_0)}\right|^*\right]$$

The functions in the previous equations are defined by Eqs. (7–10), and Theorem 1 below, and the partial derivatives $\partial\phi_k/\partial\lambda_j$, $\partial\psi_k/\partial\lambda_j$ may be obtained by integrating the adjoint system of the variational equations of Eqs. (7) and (8) backward from $\bar{t}_f$.

If Assumptions 1 and 2 are valid, then the following theorem is true.

Theorem 1 (Neighboring Optimum Guidance): if Assumptions 1 and 2 are satisfied, then there exist functions $\phi(t;t_0,x_0, \lambda_0)$, $\psi(t;t_0,x_0,\lambda_0)$, i.e., general solutions, such that i) there is a real number $\Omega > 0$ such that for each $t_0\epsilon[\bar{t}_0, \bar{t}_f]$ and for each (x_0,λ_0) which satisfy

$$\sum_{i=1}^{n} [|x_{i0} - \phi_i^*(t_0)| + |\lambda_{i0} - \psi_i^*(t_0)|] < \Omega$$

the functions $\phi(t;t_0,x_0,\lambda_0)$ and $\psi(t;t_0,x_0,\lambda_0)$ are general solutions of $\dot{x} = f(t,x,\lambda)$ and $\dot{\lambda} = g(t,x,\lambda)$, respectively, on $[\bar{t}_0,\bar{t}_f]$; ii) $\phi(t_0;t_0,x_0,\lambda_0) = x_0,\psi(t_0;t_0,x_0,\lambda_0) = \lambda_0$; iii) $\phi(t;t_0,x_0,\lambda_0)$ and $\psi(t; t_0,x_0,\lambda_0)$ are analytic functions at each $(t,t_0,x_0,\lambda_0)\epsilon S_3$ where

$$S_3 \equiv \{(t,t_0,x_0,\lambda_0): \ t\epsilon[\bar{t}_0,\bar{t}_f],t_0\epsilon[\bar{t}_0,\bar{t}_f], \sum_{i=1}^{n} [|x_{i0} - \phi_i^*(t_0)| + |\lambda_{i0} - \psi_i^*(t_0)|] < \Omega\}$$

iv) $\phi(t;t_0^*,x_0^*,\lambda_0^*) = \phi^*(t)$ and $\psi(t;t_0^*,x_0^*,\lambda_0^*) = \psi^*(t)$ for each $t\epsilon[\bar{t}_0,\bar{t}_f]$; v) the vector function $M[\bar{t}_f,\phi(\bar{t}_f;t_0^*,x_0^*,\lambda_0^*), \psi(\bar{t}_f;t_0^*,x_0^*,\lambda_0^*)] = 0$ defines implicitly the functions $\lambda_0 = \Lambda_0(t_0,x_0)$, $t_f = T_f(t_0,x_0)$ that exist and are unique, analytic functions of t_0 and x_0 in some neighborhood of (t_0^*,x_0^*).

Conditions (i–iv) define the properties of the analytic solution of Eqs. (7) and (8), and the proof is a straightforward modification of Theorem 8.2 (p. 35) in Ref. 7. Condition v), which actually defines the neighboring optimum guidance function, i.e., $\Lambda_0(t_0,x_0)$, and the cutoff equation, i.e., $T_f(t_0,x_0)$, is a consequence of classical implicit function theorems for analytic functions discussed in Ref. 8. In the next section the conclusions of Theorem 1 will be interpreted further with regard to the implementation of a neighboring optimum guidance scheme.

III. Implementation of Neighboring Optimum Guidance

Theorem 1 describes the properties of the pertinent functions involved in the optimal guidance problem. These results may then be applied in various ways to guide a space vehicle. Possible implementations will be discussed in this section. Since methods for the computation of the partial derivatives of $\Lambda_0(t_0,x_0)$ and $T_f(t_0,x_0)$ by numerical means are presented in Refs. 2–6, 9, and 10, we shall not be concerned with the problem of numerically calculating the functions discussed below.

Suppose that one has determined representations for the functions $\Lambda_0(t_0,x_0)$ and $T_f(t_0,x_0)$. Then, given a state x_0 and a time t_0, the initial Lagrange multipliers that will cause the vehicle to be transferred from x_0 to $M(t_f,x_f,\lambda_f) = 0$ are determined by $\Lambda_0(t_0,x_0)$, and the transfer time is given by $T_f(t_0,x_0)$. Since the vehicle is guided by the control commands of Eqs. (5), i.e., $u(t) = \mu\{t,\phi[t;t_0,x_0,\Lambda_0(t_0,x_0)], \psi[t;t_0,x_0,\Lambda_0(t_0,x_0)]\}$, an onboard forward integration capability is necessary. That is, the function $\Lambda_0(t_0,x_0)$ gives only the initial conditions for the optimum Lagrange multipliers. Thus, to obtain the control as a function of time, the equations of motion and Euler-Lagrange equations must be integrated forward to determine the functions $x(t) = \phi[t;t_0,x_0,\Lambda_0(t_0,x_0)]$ and $\lambda(t) = \psi[t;t_0,x_0,\Lambda_0(t_0,x_0)]$ for substitution into Eqs. (5).

In Refs. 2–4, 9, and 10 the feedback guidance function $u(t,t_0,x_0)$ is obtained directly from the analysis as a power series in t_0 and x_0. Usually only the linear terms are considered. Note that the feedback guidance function obtained by integrating a linear approximation of $\Lambda_0(t_0,x_0)$ to obtain $\phi[t;t_0,x_0,\Lambda_0(t_0,x_0)]$ and $\psi[t;t_0,x_0,\Lambda_0(t_0,x_0)]$, which are then substituted into $u(t,x,\lambda)$, is not necessarily a linear function of t_0 and x_0. Since the same amount of ground-based computation goes into the determination of the linear approximations of $\Lambda_0(t,x_0)$ and $u(t,t_0,x_0)$, it might prove advantageous to allow for forward integration in the onboard guidance scheme. Indeed, this is the case since, as we will show below, the linear $u(t,t_0,x_0)$ representation is just a first-order approximation of the feedback control that is formed by integrating the linear $\Lambda_0(t_0,x_0)$ representation.

First, let us derive the linear $u(t,t_0,x_0)$ approximation that is equivalent to the linear guidance functions of Refs. 2–4, 9, and 10. Consider Eqs. (5), i.e., $u_i = \mu_i(t,x,\lambda)$. Then, the neighboring optimum guidance function is defined by $u_i = \mu_i\{t,\phi[t;t_0,x_0,\Lambda_0(t_0,x_0)],\ \psi[t;t_0,x_0,\Lambda_0(t_0,x_0)]\}$. Assume that the functions μ_i are analytic functions of t_0 and x_0 about the point $(t_0{}^*,x_0{}^*)$. [Note that we are imposing an additional analyticity assumption to obtain a linear approximation of $u(t,t_0,x_0)$.] Then, to first-order,

$$u_i \approx \mu_i\{t;\phi[t;t_0{}^*,x_0{}^*,\Lambda_0(t_0{}^*,x_0{}^*)],\ \psi[t;t_0{}^*,x_0{}^*,\Lambda_0(t_0{}^*,x_0{}^*)]\} + \sum_{j=1}^{n}\left[\frac{\partial\mu_i}{\partial x_j}\left(\frac{\partial\phi_j}{\partial t_0} + \sum_{k=1}^{n}\frac{\partial\phi_j}{\partial\lambda_{k_0}}\frac{\partial\Lambda_{k_0}}{\partial t_0}\right) + \frac{\partial\mu_i}{\partial\lambda_j}(\ldots)\right]^* \times (t_0 - t_0{}^*) + \sum_{j=1}^{n}\sum_{l=1}^{n}\left[\frac{\partial\mu_i}{\partial x_j}\left(\frac{\partial\phi_j}{\partial x_{l0}} + \sum_{k=1}^{n}\frac{\partial\phi_j}{\partial\lambda_{k_0}}\frac{\partial\Lambda_{k_0}}{\partial x_{l0}}\right) + \frac{\partial\mu_i}{\partial\lambda_j}(\ldots)\right]^* (x_{l0} - x_{l0}{}^*) \quad (11)$$

where the zero-order term is just $u_i{}^*(t)$.

Now let us consider the guidance function that results from the integration of the equations of motion and the Euler-Lagrange equations with a linear approximation of $\Lambda_0(t_0,x_0)$. In this case the guidance function is composed of the following components:

$$u_i = \mu_i[t,\phi(t;t_0,x_0,\lambda_0{}^* + \Delta\lambda_0),\ \psi(t;t_0,x_0,\lambda_0{}^* + \Delta\lambda_0)] \quad (12)$$

where

$$\Delta\lambda_0 = \partial\lambda_0/\partial x_0|^*(x_0 - x_0{}^*) + \partial\lambda_0/\partial t_0|^*(t_0 - t_0{}^*) \quad (13)$$

Note that the true initial values of t_0 and x_0 are used to form the guidance function. To show that Eq. (11) is just a first-order approximation of Eq. (12) one need only form a Taylor series expansion of Eq. (12) about $(t_0{}^*,x_0{}^*,\lambda_0{}^*)$ after noting that Eq. (12) can be written equivalently as

$$u_i = \mu_i[t,\phi(t;t_0{}^* + \Delta t_0,x_0{}^* + \Delta x_0,\lambda_0{}^* + \Delta\lambda_0),\ \psi(t;t_0{}^* + \Delta t_0,x_0{}^* + \Delta x_0,\lambda_0{}^* + \Delta\lambda_0)] \quad (12')$$

where, of course, $\Delta t_0 = t_0 - t_0{}^*$, $\Delta x_0 = x_0 - x_0{}^*$, and $\Delta\lambda_0$ is defined by Eq. (13). Thus, the feedback guidance function of Eq. (12) will be valid in a larger region of the nominal trajectory than the guidance function of Eq. (11). This is verified numerically for a simple example in section V.

To conclude this section a brief discussion of the cutoff equation $T_f(t_0,x_0)$ will be presented. The main purpose of this equation is to determine the time when, theoretically, the terminal conditions are satisfied. Since the perturbed trajectory will probably never satisfy all of the terminal conditions with a linear neighboring optimum guidance scheme, it might be desirable to choose a cutoff condition that is a function of the current state and will closely approximate mission fulfillment (e.g., a velocity cutoff condition), or include a separate terminal guidance phase. In such cases there is no need for the $T_f(t_0,x_0)$ equation. Further, in section V it is shown that the Taylor series expansion for $T_f(t_0,x_0)$ is very slowly convergent when compared to the $\Lambda_0(t_0,x_0)$ expansion for Zermelo's problem. Thus, one should consider the possibility of avoiding the $T_f(t_0,x_0)$ equation in the application of a neighboring optimum guidance scheme.

IV. The Min-Distance Comparison Technique

In the application of neighboring optimum guidance the choice of the nominal reference state to compare with the

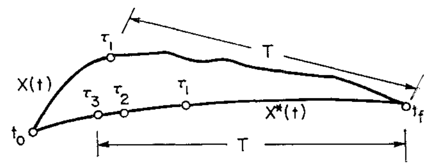

Fig. 1 Time parameters associated with optimum and near-optimum trajectories.

current (perturbed) state is not straightforward. For example, assume that $x^*(t)$, $\lambda^*(t)$ [or $u^*(t)$], and the feedback guidance gains, say $G^*(t)$, are stored onboard for each $t\epsilon[\bar{t}_0,\bar{t}_f]$ (or, for each $t_i\epsilon[\bar{t}_0,\bar{t}_f]$, $i = 1, \ldots, q$; i.e., a finite number of data points). Suppose when the clock time is equal to τ_1 that the vehicle is at the state x. A possible way of comparing this state with the nominal trajectory is to choose the values $x^*(\tau_1)$, $\lambda^*(\tau_1)$ [or, $u^*(\tau_1)$], and $G^*(\tau_1)$ for the determination of the neighboring guidance function (i.e., clocktime is the "lookup parameter"). However, $x(\tau_1)$ may not be close to $x^*(\tau_1)$ whereas $x(\tau_1)$ may be close to some other state on the nominal trajectory, say $x^*(\tau_2)$ (Fig. 1). In Refs. 9 and 10 an unpublished suggestion by J. C. Dunn is used to partially alleviate this ambiguity. In both these analyses time-to-go is used as the lookup parameter (e.g., in Fig. 1, τ_3 is the time-to-go lookup parameter when the perturbed trajectory is at τ_1 with time-to-go equal to T), and the results demonstrate that time-to-go is superior to clock time. However, to determine the time-to-go one must estimate the final time associated with the current state. This approximation depends upon the $T_f(t_0,x_0)$ equation. That is, to determine the time-to-go lookup parameter one assumes that t_0 is the clock time, $t_0{}^*$ is the lookup time, and $t_f - t_0 = t_f{}^* - t_0{}^*$. Then the following equations are solved for the unknowns t_f and $t_0{}^*$:

$$t_f = t_f{}^* + \left.\frac{\partial T_f}{\partial x_0}\right|^* [x(t_0) - x^*(t_0{}^*)] + \left.\frac{\partial T_f}{\partial t_0}\right|^* [t_0 - t_0{}^*] \quad (14)$$

$$t_f - t_0 = t_f{}^* - t_0{}^* \quad (15)$$

where $t_f{}^* = \bar{t}_f - \bar{t}_0$. Note that since $\partial T_f/\partial x_0|^*$, $\partial T_f/\partial t_0|^*$ and $x^*(t_0{}^*)$ depend upon $t_0{}^*$, an iterative scheme will probably be necessary to solve for the lookup parameter $t_0{}^*$. The solution of Eqs. (14) and (15) is eased considerably if $\partial T_f/\partial t_0|^* = 1$ (which is the case in stationary systems[10]) since then the two equations reduce to

$$\partial T_f/\partial x_0|^*[x(t_0) - x^*(t_0{}^*)] = 0 \quad (16)$$

In Ref. 11 another comparison procedure called the min-distance technique is suggested. This technique does not depend upon a $T_f(t_0,x_0)$ approximation. In addition, it does not depend upon clock time if the problem is stationary (e.g., re-entry problems). The major motivation for the method is that in many guidance missions the basic goal is to transfer the vehicle from a current state to a set of terminal conditions without regard to how the vehicle got to the current state. (For example, in a re-entry problem the current position, velocity, and orientation of the vehicle are the important quantities; the period of time that it has taken the vehicle to get to this state is not important.)

It appears that one cannot prove mathematically which comparison procedure is the best for all problems. However, as the studies in Refs. 9 and 10 show, the applicability of neighboring optimum guidance is strongly dependent upon the choice. Thus, in this section criteria for defining a comparison procedure will be suggested and then used to determine a comparison function that is problem dependent.

Given a nominal optimal trajectory that satisfies a specified mission, one can define linear (and higher-order) feedback gains based on the nominal. Let x be the current state of the vehicle, and let $x^*(t)$, $t\epsilon[\bar{t}_0,\bar{t}_f]$, be the nominal state. By

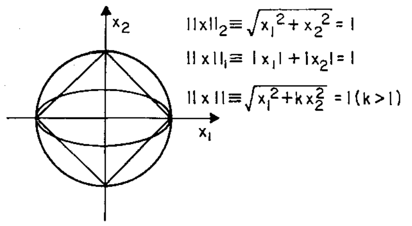

Fig. 2 The min-distance norm in 2-space.

Theorem 1, a first criterion for the comparison function is that it determines a lookup parameter which causes x to be close to a nominal state. This suggests a minimum distance comparison procedure, e.g., the parameter is defined by the value of t that minimizes the distance between x and $x^*(t)$:

$$\bar{\rho}[x,x^*(t)] = \{[x_1 - x_1^*(t)]^2 + \ldots + [x_n - x_n^*(t)]^2\}^{1/2} \tag{17}$$

This criterion is not enough, though, since it does not take into account the fact that the optimal control is, in most instances, relatively insensitive to perturbations in some of the state variables [whereas Eq. (17) treats all state perturbations equally]. Therefore, a second criterion is that the comparison function should be defined in such a way that some of the state variable perturbations have less influence than others in determining the lookup parameter. This criterion suggests a weighting procedure.

By incorporating weighting factors into Eq. (17), i.e.,

$$\rho[x,x^*(t)] \equiv \{k_1[x_1 - x_1^*(t)]^2 + \ldots + k_n[x_n - x_n^*(t)]^2\}^{1/2} \tag{18}$$

where the k_i's are sensitivity coefficients associated with perturbations in the x_i's, both criteria mentioned above may be satisfied. That is, the lookup parameter is defined by the value of t that minimizes Eq. (18). If the process is nonstationary (i.e., time appears in the right-hand sides of the equations of motion and/or the geometrical boundary conditions), then time should be treated as a state variable in Eq. (18), e.g., $x_n = t$(clock time).

Another way of viewing Eq. (18) is to consider its geometric characteristics and its relationship to the condition

$$\sum_{i=1}^{n} [|x_{i0} - \phi_i^*(t_0)| + |\lambda_{i0} - \psi_i^*(t_0)|] < \Omega \tag{19}$$

in Theorem 1. By Theorem 1, part v), the optimal Lagrange multiplier vector at a state (t_0,x_0) "close" to the perturbed trajectory is given by $\Lambda_0(t_0,x_0)$, which may be approximated to first order by

$$\Lambda_0 = \lambda_0^* + (\partial\Lambda_0/\partial x_0)|^*(x_0 - x_0^*) + (\partial\Lambda_0/\partial t_0)|^*(t_0 - t_0^*) \tag{20}$$

Actually Eq. (20) gives an indication of the sensitivity of the optimal feedback control to perturbations in x_0 and t_0. Thus, Eq. (19) may be approximated by

$$\sum_{i=1}^{n} \left[|x_{i0} - \phi_i^*(t_0)| + \left|\sum_{j=1}^{n} \frac{\partial\Lambda_{i0}}{\partial x_{j0}}\right|^* (x_{j0} - x_{j0}^*)| \right] < \Omega \tag{21}$$

where the t_0-perturbation is considered as a state perturbation, e.g., $x_n \equiv t$, if the problem is nonstationary.

In the proof of Theorem 1, the l_1 norm, i.e.,

$$||x||_1 \equiv \sum_{i=1}^{n} |x_i|$$

was used. However, the theorem may be proved with many different norms, and there may exist a better norm than the l_1 norm in the sense that if it is used to indicate an expansion point on the nominal trajectory for a perturbed state x_0, then the resultant neighboring trajectory will have better convergence characteristics. This assumption is the basis for the min-distance comparison scheme, where Eq. (18) is derived from a norm that will indicate the sensitivities within the problem.

The class of norms that generate Eq. (18) is defined by

$$||x|| \equiv \left[\sum_{i=1}^{n} k_i x_i^2\right]^{1/2}, \ (k_i \neq 0) \tag{22}$$

which defines an ellipse in 2-space, an ellipsoid in 3-space, and so on (Fig. 2). Note that in Fig. 2 the norm defined by Eq. (22) indicates that the problem is more sensitive to x_2 than x_1. Finally, note that the best set of k_i's may differ for different points on the nominal (recalling that each point on the nominal may be a possible expansion point). Thus, the ellipsoids defined by Eq. (22) may change shape along the nominal.

Let us now consider the possibility of determining a method for computing the k_i's. The first question to be answered is: Which variables do the k_i's depend upon? Since the purpose of the k_i's is to indicate the sensitivity of the optimal feedback guidance function to changes in the state variables, it follows that $k_i = k_i(x)$, i.e., the k_i's are dependent upon the state of the vehicle. As will be argued below, a deterministic method for computing the k_i's does not appear to be feasible. However, one should be able to use physical knowledge of the problem and numerical simulations of the guidance function from perturbed states about the nominal to characterize the k_i's. It appears likely that in many cases the k_i's may be suitably approximated by constants or simple functions of the state.

Suppose the vehicle is at a state x. Upon specification of the lookup parameter t_L, the neighboring optimum guidance function can be determined. Assuming that the sensitivity coefficients depend upon the state, the value of t_L is determined by solving

$$\left\{\frac{d}{dt}\sum_{i=1}^{n} k_i(x)[x_i - x_i^*(t)]^2\right\}_{t=t_L} = 0 \tag{23}$$

or,

$$\sum_{i=1}^{n} k_i(x)[x_i - x_i^*(t_L)]\dot{x}_i^*(t_L) = 0 \tag{24}$$

Since the main goal of a guidance function is satisfaction of the mission, and optimality is secondary, it is natural to choose the k_i's in such a way that a function which characterizes mission dissatisfaction is minimized. An example of such a function is the terminal miss distance. For this development suppose that we wish to choose the k_i's so that the miss distance

$$\text{Miss} = \sum_{i=1}^{p} N_i^2(t_f,x_f) \tag{25}$$

is minimized, where $N_1(t_f,x_f), \ldots, N_p(t_f,x_f)$ are the values of the specified geometrical boundary conditions at the terminal point of the perturbed trajectory. From Eq. (24), the lookup parameter can be determined as a function of the state and sensitivity coefficients, i.e.,

$$t_L = q(x_0,k) \tag{26}$$

Since the neighboring optimum guidance function is characterized by the approximations $\Lambda_i(t_L,x_0)(i = 1, \ldots, n)$, and t_f is defined by $T_f(t_L,x_0)$, Eq. (25) is strictly a function of t_L and x_0:

$$\text{Miss} = \sum_{i=1}^{p} N_i^2\{T_f(t_L,x_0),\phi[T_f(t_L,x_0);t_L,x_0,\Lambda_0(t_L,x_0)]\}, \tag{27}$$

where $x(t) = \phi(t;t_0,x_0,\lambda_0)$ is the general solution of the state.

Finally, by Eq. (26) the miss distance can be determined as a function of the perturbed state and sensitivity coefficients, i.e.,

$$\text{Miss} = \sum_{i=1}^{p} N_i{}^2(T_f[q(x_0,k),x_0],\phi\{T_f[q(x_0,k),x_0]; \quad q(x_0,k),x_0,\Lambda_0[q(x_0,k),x_0]\}) \quad (28)$$

Therefore, the sensitivity coefficients may be defined as functions of the state by minimizing Eq. (28):

$$\partial(\text{Miss})/\partial k_i = P_i(x_0,k) = 0 (i = 1, \ldots, n) \quad (29)$$

or,

$$k_i = K_i(x_0) \quad (30)$$

Even though the method described above does not appear to be feasible for the computation of the k_i's, it demonstrates that deterministic methods are conceivable. Until workable sensitivity computation methods are developed, physical insight and numerical simulations should be sufficient for the approximation of the sensitivity coefficients. For example, the k_i's might be chosen by inspecting the coefficients of the δx_{i0} terms in the variational equations, i.e.,

$$\delta\dot{x} = A(t)\delta x_0 + B(t)\delta\lambda_0(\delta x_0) = C(t)\delta x_0 \quad (31)$$

or by inspecting the coefficients in the linear approximation of the feedback control, i.e.,

$$\delta u(t) = G(t)\delta x_0 \quad (32)$$

where Eqs. (31) and (32) may be used to give estimates of allowable δx_0 perturbations if indeed the problem is analytic.

To implement the min-distance technique onboard, one may represent the nominal trajectory either by polynominals in time or by a finite number of data points. In the former case a precalculated polynomial equation in t_L [i.e., Eq. (24)] must be solved, and in the latter case a finite search for the value of t_L that minimizes Eq. (23) may be performed.

V. Simulation Results

In Ref. 9 it is shown that time-to-go is a better lookup parameter than clock time for Zermelo's problem. Since Zermelo's problem can be solved in closed-form, we shall also employ this example to demonstrate the ideas presented in the previous sections. Furthermore, we shall use the same parameter values as Kelley so that one can consult Ref. 9 for the details of the analysis.

Consider the task of guiding a boat from (0,0) to (2,1) in minimum time with boat speed $V = 1$ and a current in the z direction with velocity $p = 0.5$. The equations of motion are

$$\dot{x} = V \cos\gamma, \quad \dot{z} = p + V \sin\gamma \quad (33)$$

where the resultant nominal trajectory and control angle definition are shown in Fig. 3. In Ref. 9 the miss distance

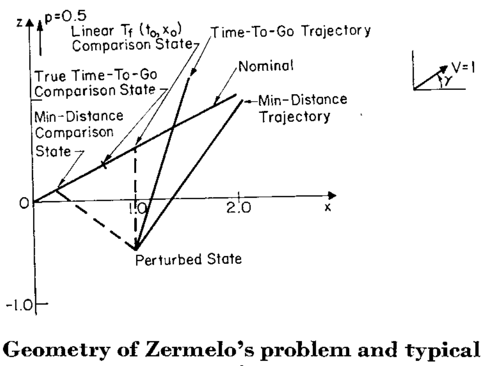

Fig. 3 Geometry of Zermelo's problem and typical trajectories.

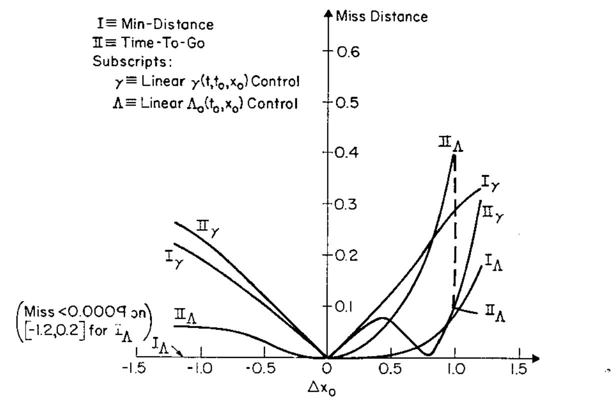

Fig. 4 Terminal miss distance with one guidance correction for state perturbations from $x_0 = 1.0$ ($z_0 \equiv 0.5$).

at cutoff was determined for various perturbations in the initial state (0,0). In that analysis a number of guidance corrections were applied between the initial perturbation and cutoff. In this section we shall consider only one guidance correction and it will be applied at the perturbed state. The reason for considering only one guidance command is that this analysis is basically concerned with determining the best possible command at a given perturbed state.

Before we consider the form of Zermelo's problem stated above, it is instructive to consider a symmetrical version with no current, i.e., Eqs. (33) with $p \equiv 0$. The optimal nominal state for this problem is $x^*(t) = 2t(5)^{1/2}$, $z^*(t) = t/(5)^{1/2}$. Since the problem is stationary, the comparison function to be minimized is

$$\rho^2 = [x - 2t/(5)^{1/2}]^2 + k_2[z - t/(5)^{1/2}]^2 \quad (34)$$

where $k_1 = 1$ can be specified arbitrarily since $d\rho^2/dt$ is a homogeneous function of the k_i's. By the symmetry of the problem one would suspect that $k_2 = 1$, also. Indeed, if one determines

$$d\rho^2/dt|_{t=t_L} = 0 \Longrightarrow t_L = (10x + 5k_2z)/[(5)^{1/2}(4 + k_2)] \quad (35)$$

and then computes $x(t)$, $z(t)$ with the linear approximation for $\Lambda_0(t_L,x,z)$, the terminal miss distance is minimized when $k_2 = 1$. Furthermore, Eq. (35) also defines the time-to-go lookup parameter for this problem, and by using this value of t_L, the exact optimum control is obtained. It is interesting to note that if the true value of time-to-go is used as the lookup parameter, the resultant miss distance is larger than with the approximate time-to-go index (or the min-distance index for this problem).

Let us now consider the unsymmetrical form of Zermelo's problem with $p = 0.5$. The optimal nominal state is given by $x^*(t) = t$, $z^*(t) = t/2$. Again the problem is stationary, and the comparison function to be minimized is

$$\rho^2 = (x - t)^2 + k_2(z - t/2) \quad (36)$$

Since there exists a current in the z direction, one would expect the problem to be more sensitive to perturbations in z (as opposed to perturbations in x). Thus, one would expect $k_2 > 1$. Also, since trajectories from perturbed states below the nominal path (Fig. 3) do not have to "fight the current" as much as trajectories from perturbed states above the nominal to meet the $z(t_f) = 1$ boundary condition, the control might be more sensitive to z perturbations above the nominal than below. If so, then $k_2(x = x^*, z > z^*) > k_2(x = x^*, z < z^*)$. In space flight guidance an analogous situation might occur with gravitational forces.

For various perturbed states, the values of k_2 that minimized the miss distance were determined. The approximate range for k_2 was $2.0 \leq k_2 \leq 2.5$, with k_2 near 2.0 only for large perturbations below the nominal (which demonstrates a decrease in sensitivity with respect to z perturbations in

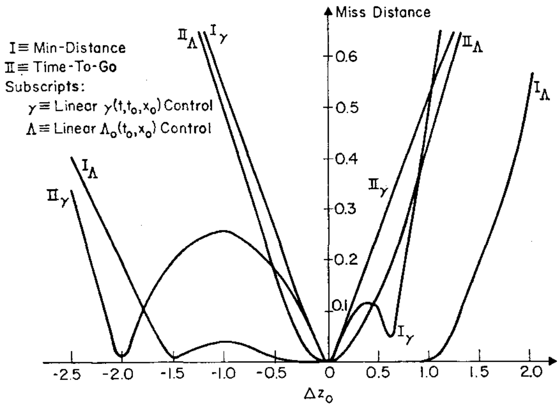

Fig. 5 Terminal miss distance with one guidance correction for state perturbations from $z_0 = 0.5$ ($x_0 \equiv 1.0$).

the negative z direction). In the immediate neighborhood of the nominal, $k_2 \approx 2.5$. Even though one can define a simple function $k_2(x,z)$ that approximately minimizes the terminal miss distance, $k_2 = 2.5$ was chosen for the example presented here since it is the most representative coefficient in a small neighborhood of the nominal. The effect of this approximation will be discussed later.

Two comparison schemes for this problem were studied: 1) min-distance with $k_2 = 2.5$, in which case the lookup parameter is

$$t_L = (4x + 2k_2z)/(4 + k_2) = (8x + 10z)/13 \quad (37)$$

and 2) time-to-go, in which case the lookup parameter is

$$t_L = x \quad (38)$$

Note that Eq. (38) can be formed by minimizing the distance function in Eq. (36) with $k_2 = 0$, and that the time-to-go lookup parameter is independent of z. This fact is contrary to intuition, and to the sensitivity analysis which emphasizes the dependence on z.

As previously stated, only one guidance correction, applied at the perturbed state, was considered. The perturbed states were defined by deviations along the $x = 1.0$ and $z = 0.5$ axes away from the nominal state (1.0, 0.5). For some large perturbations, the lookup parameters determined by Eqs. (37) and (38) were not on the interval $[\bar{t}_0, \bar{t}_f) = [0,2)$. In such cases the following rule was applied: if $t_L < 0$, then $t = 0$ was used as the index time; if $t_L \geq 2$, then $t = 1.9$ was used as the index time. [Note that if the vehicle is at $x > 2$ before cutoff, then Eq. (38) determines a negative time-to-go. Although the determination of a negative time-to-go may not be probable in a space mission, the possibility exists.]

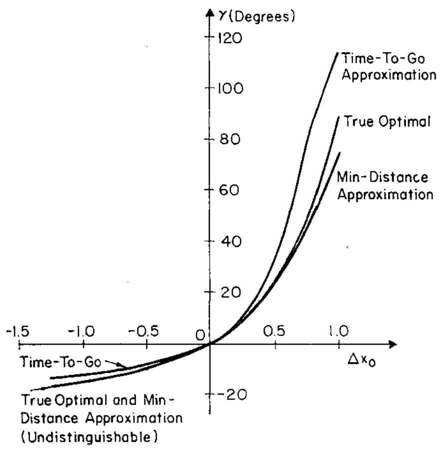

Fig. 6 Optimal control angle for state perturbations from $x_0 = 1.0$ ($z_0 \equiv 0.5$).

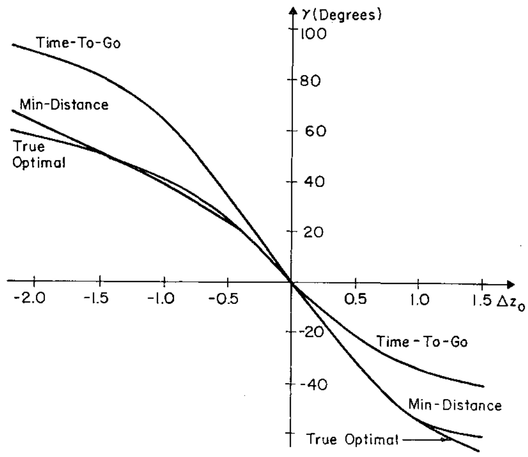

Fig. 7 Optimal control angle for state perturbations from $z_0 = 0.5$ ($x_0 \equiv 1.0$).

In Fig. 3 the resultant trajectories from the perturbed state (1.0, − 0.5) are shown. The time-to-go reference state is (1.0, 0.5) and the min-distance reference state is (0.230, 0.115). The neighboring optimum trajectory that results from the min-distance comparison technique is appreciably closer to the desired terminal conditions than the corresponding time-to-go trajectory. Also, the min-distance trajectory has a smaller miss distance than the neighboring optimum that results from the true time-to-go reference state.

In Figs. 4 and 5 the miss distances due to perturbations along the $x = 1.0$ and $z = 0.5$ axes are presented for both comparison techniques with the optimal control determined by both Eqs. (11) and (12). The miss distance varies nearly linearly with respect to state perturbations, and is less than 0.01 on the intervals [−1.0, 0.5] and [−0.5, 1.0] in Figs. 4 and 5, respectively, when the min-distance scheme with the control of Eq. (12) is used. The reason why the intervals are unsymmetrical is that $k_2 = 2.5$ was chosen. Thus, convergence would have been nearly symmetric if $k_2(x,z)$ had been utilized.

Figures 4 and 5 show that in a small neighborhood of the nominal state, the control of Eq. (12) gives better convergence than Eq. (11) for both comparison schemes. Outside of a small neighborhood the behavior becomes more erratic due to the nonlinearities in the problem. Also, neither one of the comparison schemes considered here can be classified better than the other if Eq. (11) is used as the control. If Eq. (11) is used, then Figs. 4 and 5 indicate that the sensitivity coefficient k_2 should be expressed as a function of the state with $k_2 > 1$ above the nominal and $k_2 < 1$ below the nominal. Clearly, if the control of Eq. (12) is used, the min-distance technique is appreciably better than time-to-go for both small and large perturbations. Finally, the reason why

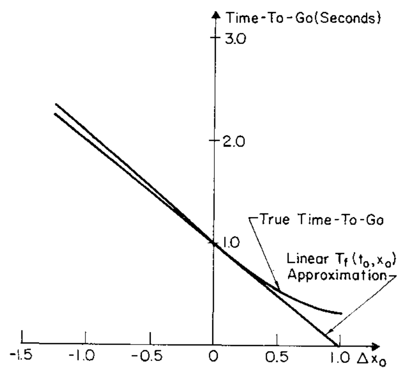

Fig. 8 Time-to-go from the perturbed state for state perturbations from $x_0 = 1.0$ ($z_0 \equiv 0.5$).

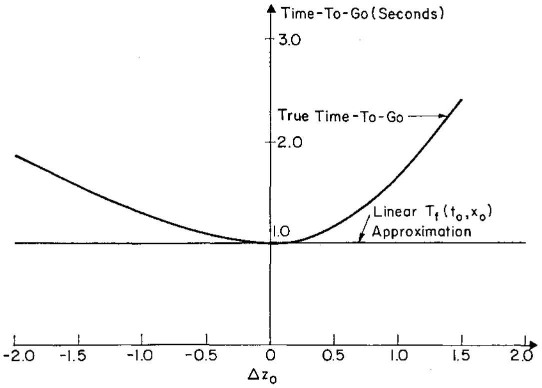

Fig. 9 Time-to-go from the perturbed state for state perturbations from $z_0 = 0.5$ ($x_0 \equiv 1.0$).

the miss distances are less in Fig. 4 than in Fig. 5 is that the perturbations in the x direction are still relatively close to the nominal even when $\Delta x_0 = \pm 1$, whereas perturbations in the z-direction result in larger distances away from the nominal as z increases and decreases away fron $z = 0.5$ at $x = 1.0$.

In Figs. 6 and 7 the control angles for the true optimal, time-to-go comparison trajectory, and min-distance comparison trajectory are shown. In all cases the min-distance approximation is closer to the optimal control than the time-to-go approximation.

Finally, Figs. 8 and 9 present the true time-to-go vs the approximate value obtained from the $T_f(t_0,x_0)$ equation. In Fig. 8 the convergence is acceptable for small perturbations, however as was previously noted, the Δx_0 perturbations result in states that are relatively close to the nominal. In Fig. 9 the convergence is unacceptable since the approximation is insensitive to perturbations in z_0 [note Eq. (38) or Eq. (50) of Ref. 9]. Thus, in the application of a neighboring optimum guidance scheme employing time-to-go, one should perform at least a numerical check of the convergence properties of the $T_f(t_0,x_0)$ expansion. Of course, if the min-distance comparison technique is used, then the $T_f(t_0,x_0)$ equation is avoided.

VI. Concluding Remarks

This analysis was concerned with the development and clarification of techniques that improve the convergence of neighboring optimum guidance. Two major aspects were considered: 1) the relationship between the linear approximation of the optimal control and the optimal control determined by the linear approximations of the Lagrange multipliers, and 2) the development of criteria for defining the comparison procedure in the application of neighboring optimum guidance. It was shown that the linear approximation of the optimal control is just the linear approximation of the control determined by the linear approximation of the Lagrange multipliers. Also, a method for comparing the perturbed state with the nominal, which minimizes a weighted distance function, was developed. A simple example was used to study the developments, and it was found that the min-distance comparison technique with the optimal control determined by the linear approximations of the Lagrange multipliers enlarged considerably the neighborhood of convergence about the nominal trajectory.

Although the analysis does not prove mathematically that the min-distance technique is better than time-to-go, it does emphasize that the choice of the comparison technique is crucial to the scheme, and that a sensitivity analysis may lead to a near optimum choice for the comparison function.

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